University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

Solution Set 3 (compiled by Daniel Larson)

1. **Griffiths 3.1** We want to calculate the average potential on the surface of a sphere due to a point charge q located somewhere within the sphere. Define our coordinates so that the sphere of radius R is centered at the origin and the point charge lies on the z-axis a distance z from the origin. This calculation is identical to the one on page 114 of the text, except that $z < R$, so when it comes time to evaluate the integral we will get a term $\sqrt{(z-R)^2} = |z-R| = R-z$.

   At any point on the sphere, the potential is $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$ where $r^2 = R^2 + z^2 - 2Rz\cos\theta$ (see Figure 3.3 for the setup, but imagine $z < R$). We need to calculate

$$\begin{aligned} V_{\text{ave}} &= \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int\frac{R^2\sin\theta\, d\theta\, d\phi}{\sqrt{z^2+R^2-2Rz\cos\theta}} = \frac{1}{4\pi\epsilon_0}\frac{q}{2Rz}\left.\sqrt{z^2+R^2-2Rz\cos\theta}\right|_0^\pi \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{2Rz}\left(\sqrt{z^2+R^2+2Rz}-\sqrt{z^2+R^2-2Rz}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{2Rz}\left(\sqrt{(z+R)^2}-\sqrt{(z-R)^2}\right) \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{2Rz}(z+R-(R-z)) = \frac{q}{4\pi\epsilon_0 R} \end{aligned}$$

   Note the term $R-z$ as mentioned above, since $z < R$ for a charge inside the sphere. Also notice that the above result doesn't depend on the exact location of the point charge. Thus if there were more than one charge, we would find $V_{\text{ave}} = \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R}$. Putting this together with the result in the text for charges outside the sphere, we have

$$V_{\text{ave}} = V_{\text{center}} + \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R}$$

2. **Griffiths 3.4** We have a region of space enclosed by one or more boundaries, the charge density ρ is given inside the region, and *either* V or $\frac{\partial V}{\partial n}$ is specified on each boundary. (The situation is much like Figure 3.6 in the text, but we're not assuming any surface is a conductor.) To prove that a solution is unique, we assume that there are two different solutions and then show that they must be equal. So assume that there are two different electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ in the region that satisfy

$$\boldsymbol{\nabla}\cdot\mathbf{E}_1 = \frac{\rho}{\epsilon_0} = -\nabla^2 V_1 \qquad\qquad \boldsymbol{\nabla}\cdot\mathbf{E}_2 = \frac{\rho}{\epsilon_0} = -\nabla^2 V_2$$

   Now let $\mathbf{E}_3 = \mathbf{E}_1 - \mathbf{E}_2$ and $\mathbf{E}_3 = -\boldsymbol{\nabla}V_3 = -\boldsymbol{\nabla}(V_1 - V_2)$. Subtracting the above equations we find $\boldsymbol{\nabla}\cdot\mathbf{E}_3 = \boldsymbol{\nabla}\cdot\mathbf{E}_1 - \boldsymbol{\nabla}\cdot\mathbf{E}_2 = 0$. Then

$$\boldsymbol{\nabla}\cdot(V_3\mathbf{E}_3) = V_3(\boldsymbol{\nabla}\cdot\mathbf{E}_3) + \mathbf{E}_3\cdot(\boldsymbol{\nabla}V_3) = \mathbf{E}_3\cdot(-\mathbf{E}_3) = -(E_3)^2$$

   Now using the divergence theorem on the above equation for a surface S_i that encloses a volume $\mathcal{V}_i$, we have:

$$\int_{S_i} V_3\mathbf{E}_3\cdot d\mathbf{a} = -\int_{\mathcal{V}_i}(E_3)^2\, d\tau. \tag{1}$$

   Now there are two cases. (I) If the potential is specified on the surface S_i, then we must have the two different potentials agree there, namely $V_1(S_i) = V_2(S_i)$, which means

$$\int_{S_i} V_3\mathbf{E}_3\cdot d\mathbf{a} = \int_{S_i}(V_1 - V_2)\mathbf{E}_3\cdot d\mathbf{a} = 0.$$

   (II) If the normal derivative $\frac{\partial V}{\partial n} = \boldsymbol{\nabla}V\cdot\hat{\mathbf{n}}$ is specified on the surface S_i, then we must have $\frac{\partial V_1}{\partial n}(S_i) = \frac{\partial V_2}{\partial n}(S_i)$, so

$$\int_{S_i} V_3\mathbf{E}_3\cdot d\mathbf{a} = -\int_{S_i} V_3\boldsymbol{\nabla}(V_1 - V_2)\cdot\hat{\mathbf{n}}da = -\int_{S_i} V_3(\boldsymbol{\nabla}V_1\cdot\hat{\mathbf{n}} - \boldsymbol{\nabla}V_2\cdot\hat{\mathbf{n}})da = -\int_{S_i} V_3\left(\frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n}\right)da = 0.$$

But looking back at equation (1) above, we see that both cases imply

$$0 = \int_{\mathcal{V}_i} (\mathbf{E}_3)^2 \, d\tau = \int_{\mathcal{V}_i} |\mathbf{E}_1 - \mathbf{E}_2|^2 .$$

If we do the integral over all the surfaces in the region, the volume $\mathcal{V}_i$ is simply the total volume of the region. Since the integrand, $|\mathbf{E}_1 - \mathbf{E}_2|^2 \geq 0$, the only way the above equation can hold is if the integrand is in fact equal to zero, which means $\mathbf{E}_1 = \mathbf{E}_2$. Thus the field is uniquely determined if the charge density is given everywhere and either V or $\frac{\partial V}{\partial n}$ is specified on each boundary.

3. **Griffiths 3.6** The xy plane is a grounded conductor, so it is at zero potential. We can reproduce this situation by considering a similar setup without the conductor, but instead with a charge $+2q$ at $z = -d$ and a charge $-q$ at $z = -3d$. These image charges make the potential $V = 0$ anywhere in the xy plane, so it exactly matches the boundary conditions in the original problem with the conductor. The force on the charge $+q$ is then given by Coulomb's Law:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\left(\frac{(q)(-q)}{(6d)^2} + \frac{(q)(2q)}{(4d)^2} + \frac{(q)(-2q)}{(2d)^2}\right)\hat{\mathbf{z}} = \frac{1}{4\pi\epsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{\mathbf{z}}.$$

4. **Griffiths 3.9** Again, we want to find some image charges that give $V = 0$ in the xy plane. So we put a uniform line charge $-\lambda$ parallel to the x-axis and a distance d directly *below* it.

   (a) The potential due to a single infinite line charge is $V(r) = -\frac{2\lambda}{4\pi\epsilon_0}\ln(r/r_0)$ where r is the perpendicular distance to the line charge and r_0 is an arbitrary reference distance. Let's choose the reference distance to be d for both the positive and negative line charges; this automatically gives zero potential on the xy-plane. We want to find the potential at an arbitrary point in the yz plane (the potential must be independent of x because of translational symmetry in the x-direction). Let s_+ and s_- be the perpendicular distance between the point $P = (y, z)$ and the positive and negative line charges. The potential at P is the sum of the potentials due to each line charge:

$$V(y,z) = \frac{2\lambda}{4\pi\epsilon_0}\left(\ln\frac{s_-}{d} - \ln\frac{s_+}{d}\right) = \frac{2\lambda}{4\pi\epsilon_0}\ln\frac{s_-}{s_+} = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{s_-^2}{s_+^2} = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right)$$

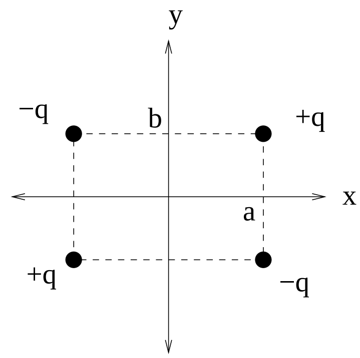
   We can check our result by verifying that $V(z=0) = 0$ as it must, since the conductor in the xy plane is grounded.

   (b) To find the charge density on the conducting plane of the original problem, we make use of Equation 2.49. In this case the normal to the xy plane is in the z direction.

$$\sigma(y) = -\epsilon_0 \frac{\partial V}{\partial n}\bigg|_{z=0} = -\epsilon_0\frac{\partial V}{\partial z}\bigg|_{z=0} = -\epsilon_0\frac{\lambda}{4\pi\epsilon_0}\left(\frac{2(z+d)}{y^2+(z+d)^2} - \frac{2(z-d)}{y^2+(z-d)^2}\right)\bigg|_{z=0} = -\frac{\lambda d}{\pi(y^2+d^2)}$$

5. **Griffiths 3.10** We want to find the potential in the first quadrant, so we are only allowed to add image charges outside this region. We can add an image charge $-q$ at $(x, y) = (a, -b)$ to give zero potential along the x-axis. To get zero potential along the y-axis we need to add *two more* image charges to balance the two charges we have already. They should have opposite charge and be placed as shown in the figure below. Assume all the charges lie in the xy plane. The potential is the sum of the contributions from the four charges:

$$\begin{aligned} V(x,y,z) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-a)^2+(y-b)^2+z^2}} + \frac{q}{\sqrt{(x+a)^2+(y+b)^2+z^2}}\right. \\ &\qquad \left. - \frac{q}{\sqrt{(x+a)^2+(y-b)^2+z^2}} - \frac{q}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right] \end{aligned}$$



Problem 5. Griffiths 3.10

The force on q due to the conducting planes is the same as the force on q due to the image charges, which is a sum of three contributions. But we need to remember that the force is a vector and keep track of all three components. First of all, since the charges all lie in the xy plane, there is no z-component: $F_z = 0$. The other components follow from Coulomb's law and breaking the force vectors into components.

$$F_x = \frac{1}{4\pi\epsilon_0}\left[\frac{-q^2}{4a^2} + \frac{q^2}{4(a^2+b^2)}\frac{a}{\sqrt{a^2+b^2}}\right] \qquad F_y = \frac{1}{4\pi\epsilon_0}\left[\frac{-q^2}{4b^2} + \frac{q^2}{4(a^2+b^2)}\frac{b}{\sqrt{a^2+b^2}}\right]$$

The easiest way to find the work needed to bring the charge q in from infinity into the corner made by the conducting planes is to compute the total work needed to bring together the collection of image charges and then divide by 4, because we don't count the work needed to bring in the image charges, for in the original problem the only other charges present are those induced in the conductors, but the induced charge comes "for free" because conductors are equipotential surfaces. Thus the work to bring in the single charge q is (using equation 2.40 and multiplying by 1/4):

$$W = \frac{1}{4}\frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{2a} - \frac{q^2}{2b} + \frac{q^2}{2\sqrt{a^2+b^2}} - \frac{q^2}{2a} - \frac{q^2}{2b} + \frac{q^2}{2\sqrt{a^2+b^2}}\right) = -\frac{q^2}{16\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{\sqrt{a^2+b^2}}\right)$$

This method would work for any angle which evenly divides 360°, namely $360°/2n$ for $n = 1, 2, 3, \ldots$.

6. **Griffiths 3.14**

(a) In this problem, the pipe is infinite in the z-direction, so there can be no dependence on z because of the symmetry. Thus we are left with solving Laplace's equation in two dimensions:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Using separation of variables, we assume $V(x, y) = X(x)Y(y)$. Plugging into the above equation, and letting primes denote derivatives of single variable functions with respect to their argument, we get

$$Y(y)X''(x) + X(x)Y''(y) = 0 \quad \Rightarrow \quad \frac{1}{X}X''(x) + \frac{1}{Y}Y''(y) = 0$$

In order fo this equation to hold for all x and y, we must have both terms equal to constants. Since the potential must vanish at $y = 0$ and $y = a$, it makes sense to use sines and cosines in the y-direction, which means we want to put a negative constant in the y-equation.

$$\frac{1}{Y}Y''(y) = -k^2 \qquad \frac{1}{X}X''(x) = k^2 \quad \text{for } k \text{ constant}$$

The solutions for the y equation give $Y(y) = A\sin ky + B\cos ky$. For the x equation, we need it to vanish at $x = 0$, so lets choose hyperbolic trig functions instead of exponentials: $X(x) = C\sinh kx + D\cosh kx$.

Thus $V(x,y) = (A\sin ky + B\cos ky)(C\sinh kx + D\cosh kx)$. Now we need to choose the coefficients A, B, C, D to satisfy the boundary conditions. $V(x, y=0) = 0$ means we need $B = 0$. $V(x=0, y) = 0$ means we need $D = 0$. $V(x, y=a) = 0$ means we need $\sin ka = 0 \;\Rightarrow\; ka = n\pi$ for $n = 1, 2, 3, \ldots$. The most general solution at this stage is a linear combination of solutions for different n, where I've combined the constants A and C into A_n:

$$V(x,y,z) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

To determine the A_n we need the last boundary condition, $V(x=b, y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right) = V_0(y)$. Using Fourier's trick, we multiply both sides by $\sin\left(\frac{m\pi y}{a}\right)$ and integrate from 0 to a. This gives

$$A_m \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{m\pi y}{a}\right)dy \quad\Rightarrow\quad A_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^\infty V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

Thus the equation for $V(x,y,z)$ together with the formula for A_n gives a general formula for the potential within the pipe.

(b) With $V_0(y) = V_0 =$ constant, we find

$$A_n = \frac{2}{a\sinh(n\pi b/a)}V_0\int_0^\infty \sin\left(\frac{n\pi y}{a}\right)dy = \frac{2V_0}{a\sinh(n\pi b/a)}\begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{2a}{n\pi}, & \text{if } n \text{ is odd.}\end{cases}$$

So we find

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}\frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}.$$

7. **Griffiths 3.15** Another problem where we need to use separation of variables, but this time with all three dimensions. Proceeding as before, we assume $V(x,y,z) = X(x)Y(y)Z(z)$ and plug this into Laplace's equation, to find

$$\frac{1}{X}X''(x) + \frac{1}{Y}Y''(y) + \frac{1}{Z}Z''(z) = 0.$$

Each of these terms must be constant, and the sum of the three constants must be zero. We want to choose the constants appropriately by looking at the boundary conditions. In the x and y directions there are grounded plates at 0 and a, which means the solutions will be sines and cosines in those directions, so we choose the constants for the X and Y terms to be negative. In order to add to zero, the other constant must be positive.

$$\frac{1}{X}X''(x) = -k^2 \qquad \frac{1}{Y}Y''(y) = -l^2 \qquad \frac{1}{Z}Z''(z) = k^2 + l^2 \qquad \text{for } k,\ l \text{ constants}$$

Now we can write down the general solutions to these equations. Since Z must vanish at $z = 0$, it is easier to write down the solution in terms of hyperbolic trig functions instead of real exponentials.

$$X(x) = A\sin kx + B\cos kx, \quad Y(y) = C\sin ly + D\cos ly, \quad Z(z) = E\sinh(z\sqrt{k^2+l^2}) + F\cosh(z\sqrt{k^2+l^2})$$

The boundary conditions tell us $V(0,y,z) = V(x,0,z) = V(x,y,0)$, so to make this hold for all values of the other variables, we must have $B = D = F = 0$. Then $V(a,y,z) = V(x,a,z) = 0$ requires $k = n\pi/a$ and $l = m\pi/a$ for positive integers n and m. So at this stage, the most general solution is a linear combination of solutions for all n and m.

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\pi z\sqrt{n^2+m^2}}{a}\right)$$

The only thing that remains is to fix the constants $A_{n,m}$ by using the last boundary condition: $V(x,y,a) = V_0$. Using Fourier's trick, we set $z = a$, multiply both sides by $\frac{2}{a}\sin(n'\pi x/a)\frac{2}{a}\sin(m'\pi y/a)$ and integrate over both x and y from 0 to a. This will pick out the coefficient $A_{n',m'}$.

$$A_{n',m'}\sinh(\pi\sqrt{n'^2+m'^2}) = \left(\frac{2}{a}\right)^2 V_0\int_0^a\int_0^a \sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{a}\right)dx\,dy = \begin{cases} 0, & \text{if } n' \text{ or } m' \text{is even,} \\ \frac{16V_0}{\pi^2 n'm'}, & \text{if both are odd.}\end{cases}$$

The above equation gives us $A_{n,m}$, which we can plug into the double sum above. The final solution is

$$V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{\text{odd } n} \sum_{\text{odd } m} \frac{1}{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \frac{\sinh(\pi z\sqrt{n^2+m^2}/a)}{\sinh(\pi\sqrt{n^2+m^2})}.$$

8. **Griffiths 3.43**

(a) To use the hint, we need to figure out what it means to integrate by parts in three dimensions. We can work it out starting from vector identity (5) in the front cover of Griffiths. With a scalar function V and a vector function $\mathbf{E}$ the identity can be written

$$\mathbf{E}\cdot(\boldsymbol{\nabla}V) = \boldsymbol{\nabla}\cdot(V\mathbf{E}) - V(\boldsymbol{\nabla}\cdot\mathbf{E}).$$

Now we integrate both sides over a volume $\mathcal{V}$ with surface S and use the divergence theorem on the first term on the right hand side. This yields a formula for three-dimensional integration by parts:

$$\int_{\mathcal{V}} \mathbf{E}\cdot(\boldsymbol{\nabla}V)\,d\tau = \int_S V\mathbf{E}\cdot d\mathbf{a} - \int_{\mathcal{V}} V(\boldsymbol{\nabla}\cdot\mathbf{E})\,d\tau$$

Now assume we have two completely different systems, numbered 1 and 2, each of which has a certain charge density ρ, potential V and electric field $\mathbf{E}$. Following the hint we will integrate $\mathbf{E}_1\cdot\mathbf{E}_2$ in two ways.

$$\int_{\mathcal{V}} \mathbf{E}_1\cdot\mathbf{E}_2\;d\tau = -\int_{\mathcal{V}}(\boldsymbol{\nabla}V_1)\cdot\mathbf{E}_2\,d\tau = -\int_S V_1\mathbf{E}_2\cdot\,d\mathbf{a} + \int_{\mathcal{V}} V_1(\boldsymbol{\nabla}\cdot\mathbf{E}_2)\,d\tau = -\int_S V_1\mathbf{E}_2\cdot\,d\mathbf{a} + \int_{\mathcal{V}} V_1\rho_2/\epsilon_0\,d\tau$$

If we assume that the charge distributions are localized (i.e. do not extend to infinity) then we can take our volume to be all of space, which means that the surface S is at infinity, where the potential V_1 falls off to zero. So the surface integral vanishes, leaving

$$\int_{\mathcal{V}} \mathbf{E}_1\cdot\mathbf{E}_2\;d\tau = \frac{1}{\epsilon_0}\int_{\mathcal{V}} V_1\rho_2\,d\tau$$

We can do exactly the same manipulations after replacing $\mathbf{E}_2$ with $-\boldsymbol{\nabla}V_2$, so we'll arrive at the same result with the labels 1 and 2 switched. So we conclude

$$\epsilon_0\int_{\mathcal{V}} \mathbf{E}_1\cdot\mathbf{E}_2\;d\tau = \int_{\mathcal{V}} V_1\rho_2\,d\tau = \int_{\mathcal{V}} V_2\rho_1\,d\tau$$

(b) Now we want to apply the above result to a specific situation. It will be less confusing if I call the conductors a and b instead of 1 and 2. In the first system we have two conductors and we put a charge Q on conductor a, and let V_{ab} be the potential at conductor b. So in this system ρ_1 is zero everywhere except on conductor a, where there is total charge Q distributed in some complicated way. But this means that $\int_{\mathcal{V}}\rho_1\,d\tau = Q$. The potential in this system is complicated. The only place we know what it is is on conductor b, where V_1 is a constant, $V_1 = V_{ab}$.

Now consider the second system. It consists of the same two conductors a and b in the same positions, but this time we put charge Q on conductor b and call the potential at conductor a V_{ba}. Here, ρ_2 is zero everywhere except on conductor b. But we know $\int_{\mathcal{V}}\rho_2\,d\tau = Q$. The potential is complicated, and all we know is that on conductor a it is constant and equal to V_{ba}.

Now we apply Green's reciprocity theorem. When we calculate $\int\rho_1 V_2\,d\tau$, ρ_1 vanishes everywhere except conductor a, but that is exactly where we know what V_2 is; it's a constant equal to V_{ba}. Thus

$$\int\rho_1 V_2\,d\tau = V_{ba}\int\rho_1\,d\tau = V_{ba}Q.$$

But we also have

$$\int\rho_2 V_1\,d\tau = V_{ab}\int\rho_2\,d\tau = V_{ab}Q,$$

where again we could do the integral because ρ_2 is zero everywhere except on conductor b where $V_1 = V_{ab}$. The Q's cancel, leaving us with the result $V_{ab} = V_{ba}$, or in the notation of the problem, $V_{12} = V_{21}$.